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# CONSISTENCY IN THE p-th MEAN OF THE LEAST SQUARES ESTIMATORS IN LINEAR MODELS

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## 1. Introduction

Let $Y_n$, $n\geq1$, and $\epsilon_n$, $n\geq1$, be two sequences of random variables satisfying

$$Y_i = x_{i1}\beta_1+x_{i2}\beta_2+\ldots+x_{iq}\beta_q+\epsilon_i \ , \ i\geq1,$$

where $x_{ij}$ , $i\geq1$, $j=1,2,\ldots,q$ are known constants, $\beta^T = (\beta_1,\beta_2,\ldots,\beta_q) \in R^q$ is the uknown parameter vector and q is a fixed positive integer. Let $Z_n^T = (Y_1,Y_2,\ldots,Y_n)$ and $X_n=(x_{ij})$, $1\leq i\leq n$ , $1\leq j\leq q$ , $n\geq1$.

If $E\epsilon_i=0$ , $i=1,2,\ldots$, then it is known that $b_n = X_n^+Z_n$ , where $X_n^+$ is the Moore-Penrose inverse of $X_n$ , will be the least squares estimator of $\beta$ based on the first n random variables $Y_1,Y_2,\ldots,Y_n$.

The problem about consistency in the p-th mean of the least squares estimators in the linear model framework has been considered in the literature. See Drygas [2,Theorem 4.2], [3,theorem 3.1] and Kaffes and Rao [4,Proposition 3.4]. Drygas considered the case p=2 in the above framework and proved that, under some conditions conserning the second moments of the $\epsilon_i$'s and the matrix $X_n$, quadratic consistency and weak consistency of $b_n$, $n\geq1$, are equivalent. Kaffes and Rao gave a necessary condition for $l^Tb_n$ , $n\geq1$, to be a sequence of consistent in the quadratic mean estimators of the estimable function $l^T\beta$ in the above framework under the assumption that $\epsilon_n$, $n\geq1$, is an equicorrelated sequence of error random variables.

In this paper we establish some results on consistency in the p-th mean ($L^p$-consistency) of least squares estimators in linear models for $0<p<\infty$.

In the sequel we need the following definitions.

Definition 1.1. Let $\beta$ be a real parameter, $b_n$ , $n\geq1$ , be a sequence of estimators of $\beta$ and $0<p<\infty$. We say that $b_n$, $n\geq1$ , is an $L^p$-consistent sequence of estimators of $\beta$, if and only if

$$\lim_{n\to\infty} E|b_n-\beta|^p=0$$

Definition 1.2. Let $\beta^T = (\beta_1,\beta_2,\ldots,\beta_q) \in R^q$ be a vector of parameters, $b_n$, $n\geq1$ , be a sequence of estimators of $\beta$, $b_n^T = (b_{n1},b_{n2},\ldots,b_{nq})$ and $0<p<\infty$.
We say that $b_n$, $n\geq1$ , is an $L^p$-consistent sequence of estimatos of $\beta$, if and only if $b_{ni}$, $n\geq1$ ,is an $L^p$-consistent sequence of estimators of $\beta_i$ ,for all $i=1,2,\ldots,q$.

## 2. Main results.

Let G be the linear manifold spanned by the vectors $(x_{n1}, x_{n2}, \ldots, x_{nq})$, $n \geq 1$, and $\epsilon^{(n)T} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$, where the $\epsilon_i$'s are conforming to the set up described above.

Theorem 2.1. Let $\epsilon_n$, $n \geq 1$, be a sequence of error random variables uniformly dominated in probability by a random variable $\epsilon$, i.e.,

$$P(|\epsilon_n| \geq a) \leq P(|\epsilon| \geq a), \text{ for every } a \geq 0 \text{ and } n \geq 1.$$

Assume further that $E|\epsilon|^p < \infty$ for some $0 < p \leq 1$ Let $1^T \in G$ and

$$1^T X_n^+ = (w_1^{(n)}, w_2^{(n)}, \ldots, w_n^{(n)}), \; n \geq 1 \; . \; If$$

$$\lim_{n \to \infty} [n . \max_{1 \leq j \leq n} |w_j^{(n)}|^p] = 0$$

then $1^T b_n = 1^T X_n^+ Z_n$, $n \geq 1$, is a sequence of $L^p$-consistent estimators of $1^T \beta$.

Proof. Since $1^T \in G$, there exists a vector $c \in R^n$ such that $1^T = c^T X_n$. Then

$$1^T b_n = 1^T X_n^+ (X_n \beta + e^{(n)}) = 1^T X_n^+ X_n \beta + 1^T X_n^+ e^{(n)} = c^T X_n X_n^+ X_n \beta + 1^T X_n^+ e^{(n)}$$

$$= c^T X_n \beta + 1^T X_n^+ e^{(n)} = 1^T \beta + 1^T X_n^+ e^{(n)} .$$

Now we observe that

$$E|1^T b_n - 1^T \beta|^p = E|1^T X_n^+ e^{(n)}|^p = E|\sum_{j=1}^n w_j^{(n)} e_j|^p$$

$$\leq E(\sum_{j=1}^n |w_j^{(n)} e_j|)^p \leq E(\sum_{j=1}^n |w_j^{(n)}|^p |e_j|^p)$$

(see Chung [1, exersise 14,p.48])

$$= \sum_{j=1}^{n} |w_j^{(n)}|^p E|\epsilon_j|^p .$$

Since the $\epsilon_i$'s are uniformly bounded in probability by the random variable $\epsilon$ ,

$$E|\epsilon_j|^p \le 1 + \sum_{n=1}^{\infty} P(|\epsilon_j|^p \ge n) \le 1 + \sum_{n=1}^{\infty} P(|\epsilon|^p \ge n) \le 1 + E|\epsilon|^p .$$

So

$$E|l^T b_n - l^T \beta|^p \le (1 + E|\epsilon|^p).(\sum_{j=1}^{n} |w_j^{(n)}|^p) \le C(n.\max_{1\le j\le n} |w_j^{(n)}|^p$$

where C is some positive constant.
This completes the proof.

Remarks 2.2. i). In theorem 2.1 we did not impose any conditions on independence of any sort on the $\epsilon_i$'s .

ii). Let $\lambda^{+}_{\min}(X_n^T X_n)$ denote the smallest positive eigenvalue of the matrix $X_n^T X_n$.If

$$\lim_{n\to\infty} (n.[\lambda^{+}_{\min}]^{-1}) = 0$$

and p=1 in the above theorem,then $l^T b_n$, n≥1, is an $L^1$-consistent sequence of estimators of $l^T\beta$ for every $l^T \in G$. For, by Cauchy-Scwartz inequality,

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n} |w_j^{(n)}| \le \sqrt{\sum_{j=1}^{n} |w_j^{(n)}|^2} \le \|l\|.\|X_n^+\| \le \|l\|.\sqrt{\frac{q}{\lambda^{+}_{\min}(X_n^T X_n)}}$$

Corollary 2.3. Let $\epsilon_n$ , n≥1 , be a sequence of error random variables uniformly dominated in probability by a random variable $\epsilon$. Assume that $E|\epsilon|^p<\infty$ for some 0<p≤1. Let $X_n^+ = (a_{ij}^{(n)})$ ,$1\le i\le q$ , $1\le j\le n$ ,$n\ge 1$ . Let Rank($X_n$) = q for some n≥q . If

$$\lim_{n\to\infty} [\max_{1\le i\le q} (n.\max_{1\le j\le n} |a_{ij}^{(n)}|^p)] = 0 ,$$

then $b_n = X_n^+ Z_n$ , $n\ge1$, is a sequence of $L^p$-consistent estimators of $\beta$.

We now examine $L^p$-consistency when $1<p<\infty$ .

Theorem 2.4. Let $\epsilon_n$, $n\ge1$, be a sequence of error random variables uniformly dominated in probability by a random variable $\epsilon$ for which we assume $E|\epsilon|^p<\infty$ for some $1<p<\infty$ . Let $l^T\in G$ and $l^T X_n^+ = (w_1^{(n)}, w_2^{(n)}, \ldots, w_n^{(n)})$, $n\ge1$. If

$$\lim_{n\to\infty} (n^p.\max_{1\le j\le n} |w_j^{(n)}|^p) = 0,$$

then $l^T b_n$, $n\ge1$ , is a sequence of $L^p$-consistent estimators of $l^T\beta$.

Proof. Since $l^T\in G$ , we can write $l^T b_n = l^T\beta + l^T X_n^+ e^{(n)}$ .
Now,

$$E|l^T b^n - l^T\beta|^p = E|\sum_{j=1}^{n} w_j^{(n)} e_j|^p \le n^{p-1}.\ E\sum_{j=1}^{n} |w_j^{(n)}|^p |e_j|^p =$$

(see Chung [1, exercise 14, p. 48])

$$n^{p-1}.\sum_{j=1}^{n} |w_j^{(n)}|^p E|e_j|^p \le (1+E|\epsilon|^p)\ n^{p-1}\sum_{j=1}^{n} |w_j^{(n)}|^p \le$$

$$Cn^{p-1}n \max_{1\le j\le n} |w_j^{(n)}|^p = C\, n^p \max_{1\le j\le n} |w_j^{(n)}|^p$$

where C is a positive constant. This completes the proof.

Remark 2.5. If in theorem 2.4 we assume $\lim_{n\to\infty} [n(\lambda_{\min}^+(X_n^T X_n))^{-1}] = 0$ for the case p=2 , we obtain that $l^T b_n$, $n\ge1$ , is a sequence of $L^2$-consistent estimators of $l^T\beta$ , for every $l^T\in G$. For

$$n\sum_{j=1}^{n} |w_j^{(n)}|^2 \le n\|l\|^2\|X_n^+\|^2 \le n\|l\|^2 q.[\lambda_{\min}^+(X_n^T X_n)]^{-1} .$$

Corrolary 2.6. Let $\epsilon_n$, $n\ge1$ , be a sequence of error random variables uniformly dominated in probability by a random variable $\epsilon$. Assume $E|\epsilon|^p<\infty$ for some $1<p<\infty$ . Let $X_n^+ = (a_{ij}^{(n)})$ , $1\le i\le q$ , $1\le j\le n$ , $n\ge1$ , and $\mathrm{Rank}(X_n) = q$ for some $n\ge q$. If

$$\lim_{n\to\infty} ( \max_{1\le i\le q} [n^p \max_{1\le j\le n} |a_{ij}^{(n)}|^p]) = 0,$$

then $b_n = X_n^+ Z_n$ , $n\ge1$ , is a sequence of $L^p$-consistent estimators of $\beta$.

3. Examples. There are no stochastic conditions on $\epsilon_n$'s which would ensure that $b_n$'s are strongly consistent whatever may be the values of the constants $x_{ij}$'s. In the following example, $\epsilon_n, n\ge1$, is a sequence of independent identically distributed normal random variables and we produce constants $x_{ij}$'s for which the sequence $b_n$, $n\ge1$ , fail to converge in probability to $\beta$, which implies that $b_n$'s are not strongly consistent.

Example 3.1.

$$Y_1 = \beta_1+\beta_2+\epsilon_1$$
$$Y_i = \beta_1+2\beta_2+\epsilon_i \quad , i\ge2 .$$

The least squares estimator $b_{n1}$ of $\beta_1$ based on $Y_1, Y_2, \ldots, Y_n$ is

$$b_{n1} = \frac{4n-3}{n-1}\sum_{i=1}^{n} Y_i + \frac{1-2n}{n-1}[Y_1+\sum_{i=2}^{n} Y_i]$$

and

$$var(b_{n1}) = \sigma^2\frac{4n-3}{n-1}$$

for $n\ge2$.
If $b_{n1}$, $n\ge1$, were to converge in probability to $\beta_1$,then $var(b_{n1}), n\ge2$, should converge to 0.But this is not the case.

There are cases in which the conditions imposed for strong consistency are met. Here is an example.

Example 3.2. Let $Y_i = \sqrt{i}\beta + e_i$ , $i\ge1$, where the $\epsilon_i$'s are uniformly bounded in probability by a random variable $\epsilon$,for which $E|\epsilon|^{3/4}<\infty$ . For this linear model identify

$X_n^T = (\sqrt{1}, \sqrt{2}, \ldots, \sqrt{n})$ of order 1xn and $a_{1j}^{(n)} = \dfrac{2\sqrt{j}}{n(n+1)}$.

It is true that

$$n.\max_{1 \le j \le n} |a_{1j}^{(n)}|^p = \frac{2^p n^{(p+2)/2}}{n^p (n+1)^p} \le 2^p n^{-(3p-2)/2} = 2^{3/4} n^{-1/8}$$

which converges to 0, as $n \to \infty$.
So the least squares estimator of $\beta$, based on the first n observations $Y_1, Y_2, \ldots, Y_n$ i.e. $b_n = \dfrac{2}{n(n+1)} \sum_{j=1}^{n} \sqrt{j}\, Y_j$ is $L^{3/4}$-consistent.

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