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Creator: HDML

BULLETIN OF THE
GREEK MATHEMATICAL SOCIETY
VOLUME 42, 1999 (43 – 48)

REMARKS ON THE COMPOSITE *G*-VALUATIONS AND THEIR HYPERMETRIC PROPERTY

by

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Introduction: It is very useful to phrase ideal theoretical problems in terms of an ordered group. The natural extension of the notion of Krull valuation has been the notion of semi-valuation, which is a useful tool in a lot of cases, and an interesting and extensive literature has been developed about the subject. The possibility of solving the problem in the partially ordered group and then pulling back the solution to the integral domain is a good stimulus for all this work.

The crucial point of all this process is the meaning which we ascribe to the hypermetric property. In semi-valuations this remains in the form $v(x+y) \geq \inf_G(v(x), v(y))$, although in some cases this leads to some unfamiliar results, especially when $v(x+y)$ becomes smaller than both $v(x)$ and $v(y)$.

Here we have to face the problem: does there exist another aspect of the hypermetric property, such that the value $v(x+y)$ is never smaller than $v(x)$ and $v(y)$? at the same time the images of the units are not zero in any case. The answer to the question is affirmative, but some troubles arise, mainly because the set $R = \{x : v(x) \geq 0\}$ is not a ring in general, although another, more complicated subset is a ring.

Next, we study composite valuations and the treatment of this subject gives a satisfactory approximation to what happens in the case of totally ordered groups, and perhaps it means that the assumptions and all the procedures are inherent in the nature of things. Under some changes and some adjustments, it is possible to transfer the theory of the composite semi-valuations as given by Ohm in [3], to the case of *G*-valuations. An appropriate homomorphism is introduced, the composite *G*-valuations are defined by analogy to the former ones, and similar conditions are stated under which an ordered exact sequence splits.

1991, AMS Classification 13A18, 20K30

1. Preliminaries

We begin by giving some definitions and notations.

1.1. Ordered groups. Throughout the sequel $(G, +, \leq)$ is an ordered abelian group; let G^+ be its positive cone ($0 \in G^+$, G^- is the negative cone) and $t(G)$ is the torsion part of G. If $D = \sum D_u$ is a direct sum of ordered groups D_u, then D is called the **ordered direct sum** : in set notation if $D^+ = \{d \in D \text{ , } d_u \geq 0 \text{ for all } u\}$.

If $a_0, a_1, \cdots, a_n$ are elements of G, denote by $a_0 \geq \inf_{\hat{G}}\{a_1, \cdots, a_n\}$, if and only if $a_0 > a$ for all $a < a_i$, $i \in \{1, \cdots, n\}$. Note the difference from the case of a semi-valuation, where $a_0 \geq \inf_G\{a_1, \cdots, a_n\}$ means that $a_0 \geq a$ for all $a \leq a_i$, $i \in \{1, \cdots, n\}$.

A short exact sequence of ordered groups

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{1}$$

is called **lexicographically exact** iff $B^+ = \{b \in B \text{ , } g(b) > 0 \text{ or } b \in f(A^+)\}$, where A^+ and B^+ are the positive cones of A and B, respectively. Such a sequence is then **order exact** (that is, $f(A^+) = f(A) \cap B^+$ and $g(B^+) = C^+$) and f, g are order-homomorphisms (that is, $f(A^+) \subset B^+$ and $g(B^+) \subset C^+$).

The direct sum $A \oplus C$ is called the **lexicographically direct sum** if we order it by: $(A \oplus C)^+ = \{(a, c) : c > 0 \text{ or } c = 0 \text{ and } a \geq 0\}$. We say that the lexicographically exact sequence (1) **splits lexicographically**, if there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_3} & & \\
0 & \longrightarrow & A & \xrightarrow[1]{} & A \oplus C & \xrightarrow[\pi]{} & C & \longrightarrow & 0
\end{array}$$

where 1 and π are the usual injection and projection maps, i_1, i_3 are the identity maps and i_2 is an order isomorphism.

1.2. *G*-valuations. Let K be a field, $(G, +, \leq)$ an ordered abelian group, $K^* = K \setminus \{0\}$ and $\hat{G} = G \cup \{\infty\}$, where ∞ is a new element such that $a + \infty = \infty + \infty = \infty + a = \infty$ and $a < \infty$, for any $a \in G$.

Definition. A G-valuation v is a function of K into $\hat{G}$ such that for all x, y in K :

(i) $v(x) = \infty \Leftrightarrow x = 0$ $\quad$ (ii) $v(x) = v(-x)$

(iii) $v(x \cdot y) = v(x) + v(y)$

(iv) $v(x + y) \geq \inf_{\hat{G}}\{v(x), v(y)\}$

The last property means the infimum in a concrete order-completion, the known Kurepa completion, where for any $c \in G$, if $v(x) > c$ and $v(y) > c$, then $v(x + y) > c$. It means that the element $v(x + y)$ belongs to the upper class B, of the cut (A, B) of G, where A is the set of all the elements of G, which are smaller than $v(x)$ and

$v(y)$. It is a simple consequence that $v(1) = 0$ and if the order of $v(-1)$ is different from 2, then the axioms (*i*), (*iii*) imply (*ii*). On the other hand, if the field K is finite and G is torsion free, then v is trivial (that is, $v(0) = \infty$ and for any non zero $x \in K$, $v(x) = 0$).

It is also true that the subset $v(K^*)$ is a subgroup of G (the G-value group of v), but the set $\{x : v(x) \geq 0\}$ fails in general to be a ring. However it is not difficult to show that the set $R_v = \{x : v(x) > c$, for all negative elements $c\}$ is a ring with identity, while the set $M_v = \{x : v(x) \in G^+ \setminus \{0\}\}$ is an ideal of R_v.

In fact; if $x, y \in R_v$, then $x - y \in R_v$ and it is impossible for $v(xy)$ to be incomparable to any $c < 0$, because $c - v(y) < 0$. Besides if $x \in R_v$, $y \in M_v$, it is impossible for $v(xy) < 0$ as well as $v(xy)$ to be incomparable to zero, because (for the last) in this case $v(x)$ will be incomparable to $-v(y) < 0$, absurd.

On the other hand, the following holds (the incomparable elements are called parallel) :

Proposition. Given a G-valuation w of a field K, if the positive elements of the value group are larger than the parallel to zero elements, then the set $R_w = \{x \in K : w(x) \geq 0$ or $w(x)$ is parallel to zero$\}$ is a ring and the set $M_w = \{x \in K : w(x) > 0\}$ is a maximal ideal.

Example. To begin, let k_0 be a field, X, Y indeterminates over k_0, and $k = k_0(X)$ the quotient field of the ring $k_0[X]$. Let u be the known valuation on $k_0(X)$ and $K = k_0[X,Y]$. Define a semi-valuation v of $k_0[X,Y]$ by $v(p_i Y^i + p_{i+1} Y^{i+1} + \cdots + p_{i+n} Y^{i+n}) = (u(p_i), i)$, where $p_{i+h} \in k_0(X)$, $h \in \{0, \cdots, n\}$, $p_i \neq 0$ and $i \in Z$. Consider the lexicographically direct product of the value group of u and Z. It follows that v is a G-valuation as well as a semi-valuation.

It is also simple to show that in the case where the group of the images of a function of a field which satisfies (*ii*) and (*iii*) has elements incomparable to each other, this function is a G-valuation as well as a semi-valuation.

1.3. The *G*-homomorphism. If B and C are ordered groups and g is a homomorphism of B into C, then g is said to be a ***G*-homomorphism,** if for every $b_0, b_1, \cdots, b_n$ in B, the relation $b_0 \geq \inf_B \{b_1, \cdots, b_n\}$, implies that $g(b_0) \geq \inf_C \{g(b_1), \cdots, g(b_n)\}$.

It is not difficult to prove the following :

Proposition (1) If v is a G-valuation defined on a field K, ranging over an ordered group B and if $g : B \to C$ is a G-homomorphism, then $g \circ v$ is a G-valuation.

(2) If, in the short exact sequence $(f, g) : A \to B \to C$, f and g are G-homomorphisms, then $g \circ f$ is also a G-homomorphism.

(3) If the sequence $(f, g) : A \to B \to C$ is lexicographically exact, then f is a G-homomorphism.

(4) If B and C are lattice groups, then the homomorphism $g: B \to C$ is a G-homomorphism iff g preserves the positiveness of the positive elements and moreover $g(\inf_B\{b_1,\cdots,b_n\}) = \inf_C\{g(b_1),\cdots,g(b_n)\}$, for every subset $\{b_1,\cdots,b_n\}$ of B.

Proof.

(1) It is $g \circ v(xy) = g(v(x)) + g(v(y))$, $g \circ v\left(\frac{1}{x}\right) = g(-v(x)) = -g \circ v(x)$,

$g \circ v(x+y) \geq \inf_C\{g(v(x)), g(v(y))\}$, since $v(x+y) \geq \inf_B\{v(x), v(y)\}$.

(2) Firstly the positive elements remain positive via the G-homomorphisms. Let be $c < c_0$ for each $c < \{c_1,\cdots,c_n\}$. It will be $f(c_0) > c'$, for each $c' < \{f(c_1),\cdots,f(c_n)\}$ and finally $g(f(c_0)) > c''$, for each $c'' < \{g(f(c_1)),\cdots,g(f(c_n))\}$.

(3) The conservation of the positivity of the positive elements, by the order exactness of f, reassures that it is G-homomorphism.

(4) It is enough, because for every $b_0 > c$ with $c < b_1$, $c < b_2$, $\cdots$, $c < b_n$, that is, $b_0 > \inf_B\{b_1,\cdots,b_n\}$ we have $g(b_0) \geq \inf_C\{g(b_1),\cdots,g(b_n)\}$.

Conversely, the condition does not make the $g(b_1),\cdots,g(b_n)$ coincide, hence the conservation of the positivity is necessary.

2. The composite *G*-valuations

In analogy to the notion of the composite semi-valuations, we define the composite G-valuations and after some differentiations and adjustments, we try to transfer the theory of Jack Ohm for semi-valuations to the case of arbitrary po-groups.

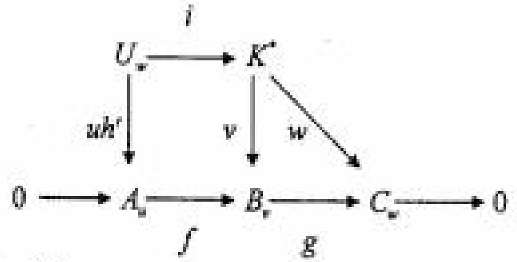
The definition of appropriate homomorphisms as well as the embedding in a concrete order-completion of the po-group, allow to us to have results analogous to those of semi-valuations.

We fix the following notation : K is always a field, w is a G-valuation of K and assume that the set R_w is a quasi-local ring with maximal ideal m_w and residue field $k = R_w/m_w$. We denote by h the canonical homomorphism of R_w into k. Let u be a G-valuation of K, and let v be any G-valuation of K such that $R_v = h^{-1}(R_u)$.

If R_u and R_v are rings, then v is said to be composite from w and u.

Let, furthermore, A_u, B_v and C_w denote the respective G-value groups of u, v and w and let U_u, U_v and U_w be the respective multiplicative groups of units of R_u, R_v and R_w.

2.1. Proposition. Suppose that R_v and R_w are rings; then there exist G-homomorphisms f and g which complete commutatively the diagram below and make the bottom row lexicographically exact (i the identity, h' the restriction of h to U_w).



(*)

The proof follows as in [3].

The definition of f and g becomes as follows :

$$\mathrm{Ker}g = \mathrm{Im}\, v\,|_{\{x\in R_w : w(x)=0\}} \text{ and } \mathrm{Ker}f = \mathrm{Im}\, uh'\,|_{U_w}$$

For x, y in K^*,

$$i(xy) = xy \xrightarrow{fuh'} v(x\cdot y) = v(x) + v(y) = fuh'(x) + fuh'(y) = f(uh'(x\cdot y)).$$

We also have $gf(A_u^+) = \{0\}$, $f(A_u^+) \subset U_w$ and $B^+ = \{x : w(x) \geq 0\}$ and $g(B_v) = C_w$.

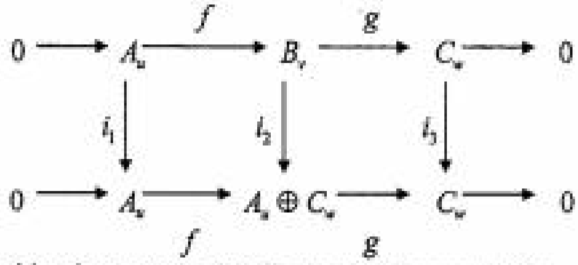
2.2. The case of C_w being a totally ordered group. In such a case R_w is a ring and given w and u we define their composition $v : K^* \to A_u \oplus C_w$ by $v(x) = (uh(x), w(x))$. Then the following is true

Proposition. If A_u is a G-value group and C_w a totally ordered group, then $A_u \oplus C_w$ is a G-value group.

Proof.

It follows from a well-known statement of Krull (cited in [3], p.31). We define a G-valuation w with value group C_w, while (by the definition of A_u) a G-valuation u is defined on the set $k = R_w/m_w$.

In that case the short exact sequence $(f, g) : A_u \to B_v \to C_w$ splits, that is



where i_1 , i_3 are the identity maps and i_2 is an order-isomorphism.

2.3. Theorem. (C.f. [2]) Let $(f, g) : A_u \to B_v \to C_w$, $A_u \neq \{0\}$ be a lexicographically exact sequence and v be a G-valuation of a field K with G-value group B_v and its assigned set R_v a local ring. Then, (1) a G-valuation w of K into C_w is defined with R_w a local ring, (2) the ideal m_w is maximal and (3) a G-valuation u of the residue field R_w/m_w is defined with G-value group A_u and for which the commutative diagram (*) is valid.

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